Theoretical Computer Science 16 (1981) 211–221
North-Holland Publishing Company

NOTE

A DUAL PROBLEM TO LEAST FIXED POINTS

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Communicated by M. Nivat
Received October 1980
Revised March 1981

Abstract. After a simple and convenient generalization of the notion of continuous functions and continuous lattices we answer the following question: when for a given element x of a complete lattice there is a least continuous function having x as a least fixed point? The minimal continuous functions having x as a least fixed point are characterized through a correspondence with maximal ascending sequences converging to x.

1. Introduction

One of the main achievements of Scott's Mathematical Semantics has been to provide an algebraic framework to define in a rigorous way constructs such as recursion [6] (different treatments are to be found in [2, 4, 7]). A recursive function x is defined as the solution of an equation of the form $x = f(x)$ where f is a continuous functional whose domain and range is a continuous lattice of functions. Tarski's theorem [8] for monotonic functions ensures that such an equation has a non-empty set of solutions with a minimum element called the least fixed point. For continuous functions a construction of the least fixed point is possible: it is the limit of a set of increasingly 'better' approximations.

In the second part simple extensions of definitions and basic properties related to lattices and continuity are given. In the third part we consider the following problem: find a canonical function among those having the same least fixed point. This problem is related to the choice of a particular recursive program to define a given function and maybe also of some relevance to people who study fixed points in the line of Manna and Shamir [5]. As we have a partial order we can consider a least element, a minimal element, a greatest element or a maximal element. We will study here the existence of least and minimal elements for the following reason: the essence and beauty of Scott's method is to consider a recursively defined function as the limit of a sequence of more and more defined functions in a lattice. The interesting case is when one defines an unknown function x as the limit of an infinite sequence of known functions. If for instance one chooses a functional greater or equal to a constant functional to recursively define x, the corresponding sequence of approximations would be finite and practically one would define x as its own approximation. Therefore, in order to obtain approximations as rich as possible and avoid missing the infinite ones when they exist, we are led to consider the existence of a least or minimal function having x as its least fixed point. The main results in the fourth part establish a one to one correspondence between the set of minimal elements having x as a least fixed point and the maximal chains converging to x.

The definitions of partial order, lattice, complete lattice, least upper bound (lub), greatest lower bound (glb), directed sets, continuous lattices and functions, monotonic functions are all standard and may be found in Birkhoff [1], de Bakker [2], Stoy [7]. An ascending sequence in a lattice D is a set of elements $\{x_i\}_{i\in N}$ such that $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant x_{n+1} \leqslant \cdots$. Some papers referring to Scott's theory call chain what we define as an ascending sequence. Here we use the original definition of a chain: a totally ordered set.

2. Lattices and σ-continuity

In the literature one finds different definitions for continuous functions related to ascending sequences, or chains or directed sets. Instead of considering separately these three cases it was decided to introduce a new definition for continuous functions which allows us to provide a single set of statements and proofs valid for the different usual notions. This definition of continuity is a particular case of the Z continuity defined in [9]. The following definitions help us present the properties of families of subsets which are relevant to our notion of continuous functions.

Remark. To simplify the notation we will write $f(S)$ instead of $\{f(d) \mid d \in S\}$, glb(x, S) instead of $\{\text{glb}(x, t) \mid t \in S\}$ etc.

Definition. A family F of subsets of a complete lattice D is a *σ-family* if and only if:

(a) all the elements of F are directed sets;
(b) $\forall S \in F$ and for any monotonic function $f: D \to D$, $f(S) \in F$,
(c) F contains all the ascending sequences in D.

Definition. Let D be a complete lattice, $d \in D$ is a *σ-limit point* with respect to some σ-family F if and only if there exists S in F, $d \notin S$, $d = \text{lub } S$. Such a set S is called an *interesting set.*

As one can easily verify the families of ascending sequences chains and directed sets are examples of σ-families in a complete lattice.

We can now introduce the notion of continuity with respect to σ-families.

Definition. Let D be a complete lattice, a function $f: D \to D$ is *σ-continuous* with respect to a σ-family F if and only if:

$$\forall S \in F: f(\text{lub } S) = \text{lub } f(S).$$

Clearly any σ-continuous function $f: D \to D$ on a complete lattice D has a least fixed point: $\text{lub}\{f^n(\perp), n \in N\}$ and a function $f: D \to D$ is σ-continuous if and only if f is monotonic and for every interesting subset S

$$f(\text{lub } S) = \text{lub}(f(S)).$$

The continuity or non-continuity of the glb operator (which can be appropriately formally defined) plays an important role in our study and leads us to the following definition and the easily derived proposition:

Definition. A complete lattice D is *σ-normal* with respect to a σ-family F if and only if

$$\forall S_1, S_2 \in F: \text{glb}(\text{lub } S_1, \text{lub } S_2) = \underset{t \in S_1}{\text{lub}} \ \underset{u \in S_2}{\text{lub}} \ (\text{glb}(t, u)).$$

Proposition 2.1. *A complete lattice is σ-normal with respect to a σ-family F if and only if*

$$\forall S \in F, \forall d \in D: \text{glb}(\text{lub } S, d) = \text{lub}(\text{glb}(S, d)).$$

Many familiar lattices are σ-normal, including the more important ones in Scott's theory: continuous lattices are σ-normal with respect to the family of directed sets. This result is a direct consequence of the following definitions and proposition:

Definition. Let F be a σ-family of complete lattice D, $x, y \in D$. Then $x \ll_\sigma y$ (x is σ-well below y) if and only if

$$\forall S \in F \text{ such that } y \leq \text{lub } S \ \exists t \in S \text{ such that } x \leq t.$$

Definition. A complete lattice D is *σ-continuous* with respect to a σ-family G if and only if

$$\forall y \in D: \text{lub}(x \mid x \ll_\sigma y) = y.$$

Proposition 2.2. *A complete lattice D, σ-continuous with respect to a σ-family F, is σ-normal.*

Proof. Assume that D is not σ-normal; there exists S in F and there exists d in D such that

$$z = \text{lub}(\text{glb}(S, d)) < \text{glb}(\text{lub } S, d) = y.$$

Consider now an element x such that $x \ll_\sigma y$. $\forall T \in F$ such that $y \leq \text{lub } T$ there exists t in T such that $x \leq t$. For $T = S$ we have $x \leq \text{glb}(t, d)$ since $x \leq y \leq d$ and $x \leq t$. So $x \leq \text{lub}(\text{glb}(S, d)) = z < y$ and $\text{lub}(x \mid x_\sigma \ll y) \leq z < y$. Therefore D is not σ-continuous.

3. Least elements of $Y_\sigma^{-1}(x)$

Let D be a complete lattice, F a σ-family of subsets of D, $[D \to D]$, the set of σ-continuous functions from D into itself and Y_σ the least fixed point operator, which associates to each σ-continuous function f its least fixed point $Y_\sigma(f) = \text{lub}\{f^n(\perp)\}$. The order on $D \to D$ is the usual: $\forall f, g \in D \to D, f \leq g$ iff $\forall d \in D\ f(d) \leq g(d)$. Let x be a given element of D, $Y_\sigma^{-1}(x)$ the set of σ-continuous functions having x as a least fixed point. We now give a series of conditions for $Y_\sigma^{-1}(x)$ to have a least element: l_x. $Y_\sigma^{-1}(x)$ is not empty as it contains the constant function $X(D) = x$, therefore the first condition is obvious:

Lemma 3.1. *If l_x exists, then $\forall y \geq x$, $l_x(y) = x$, otherwise $l_x(y) \leq x$.*

Notation. The set of elements of a lattice D, above an element x of the lattice will be written: $A(x) = \{y \in D \mid x \leq y\}$, similarly $B(x) = \{y \in D \mid y < x\}$ represents the elements strictly below x and finally $C(x) = \{y \in D \mid y \notin A(x)$ and $y \notin B(x)\}$ represents the elements that cannot be compared to x.

From the demonstration of Tarski's theorem [1] one can derive easily:

Lemma 3.2. *Let f be a monotonic function $D \to D$ where D is a complete lattice and let x be its least fixed point, then*

$$\forall d \in D: d < x \Rightarrow f(d) \not\leq d.$$

Lemma 3.3. *If l_x exists, then $\{x\} \cup B(x)$ is the ascending sequence $\{x\} \cup \{l_x^n(\perp), n \in N\}$.*

Proof. We first show that $B(x)$ is a chain and then that it is an ascending sequence. If $B(x)$ is not a chain there exist a, b in $B(x)$ such that $c = \text{glb}(a, b)$ and $c < a$, $c < b$. The function l_x must be less than the step functions g and h defined as:

$$\forall d \in D,\ g(d) = a \text{ if } d \leq c,\ x \text{ otherwise},$$

$$\forall d \in D,\ h(d) = b \text{ if } d \leq c,\ x \text{ otherwise}$$

which clearly belong to $Y_\sigma^{-1}(x)$. Now $l_x(c) \leq g(c) = a$, $l_x(c) \leq h(c) = b$ and therefore $l_x(c) \leq c$ which is not possible according to the previous lemma. If $\{x\} \cup B(x)$ is not reduced to the ascending sequence $\{x\} \cup \{l_x^n(\perp)\}$, then there exist $y \in B(x)$ and an integer i such that $l_x^i(\perp) < y < l_x^{i+1}(\perp)$. Clearly we must have $l_x \leqslant f_y$ defined as $\forall d \in D$, $f_y(d) = y$ if $d \leqslant l_x^i(\perp), x$ otherwise, which belongs to $Y_\sigma^{-1}(x)$. But $l_x(l_x^i(\perp)) = l_x^{i+1}(\perp) \not\leqslant f_y(l_x^i(\perp)) = y$. ∎

These conditions give us the values taken by l_x on $A(x)$ and $B(x)$, now we will characterize the values taken by l_x on $C(x)$.

Definition. Let $B(x)$ be an ascending sequence $\{x_n\}$, we define the function *next* on $B(x) \cup \{x\}$ as: $\text{next}(x_n) = x_{n+1}$ if x_n is not the last element of $B(x)$, $\text{next}(x_n) = x$ if x_n is the last element of $B(x)$ and finally $\text{next}(x) = x$.

The following lemma is easily verified

Lemma 3.4. *Let $B(x)$ be an ascending sequence. For every t in $C(x)$ the function h_t defined as:*

$$\forall d \in D: h_t(d) = \text{next}(\text{glb}(t, x)) \textit{ for } d \leqslant t,\ h_t(d) = x \textit{ otherwise}$$

belongs to $Y_\sigma^{-1}(x)$.

Now we are ready to characterize fully l_x when it exists.

Lemma 3.5. *If l_x exists, then it is equal to the function k_x defined as*

$$\forall d \in D: k_x(d) = \text{next}(\text{glb}(d, x)).$$

Proof. The two previous conditions have shown that $l_x(d) = x$ for $d \in A(x)$ and $l_x(d) = \text{next}(d)$ for $d \in B(x)$. Now assume $d \in C(x)$, we must have

$$l_x(d) \leqslant h_d(d) = \text{next}(\text{glb}(d, x)).$$

However

$$\text{glb}(d, x) \leqslant d \Rightarrow l_x(\text{glb}(d, x)) \leqslant l_x(d).$$

As

$$\text{glb}(d, x) \in B(x),\ l_x(\text{glb}(d, x)) = \text{next}(\text{glb}(d, x)).$$

The function k_x always exists when $B(x)$ is an ascending sequence and in that case, l_x will exist if and only if k_x is σ-continuous, that is if and only if $\forall S \in F\ k_x(\text{lub } S) = \text{lub}(k_x(S))$, as it is clear that k_x is monotonic.

Lemma 3.6. *If $B(x)$ is an ascending sequence k_x is σ-continuous if and only if*

$$\forall S \in F: S \subseteq C(x) \cup B(x), \textit{ either } \text{lub}(\text{glb}(S, x)) = \text{glb}(\text{lub } S, x)$$

or $B(x)$ is finite, $x = \text{glb}(\text{lub } S, x)$, and $\text{lub}(\text{glb}(S, x))$ is equal to the greatest element of $B(x)$.

Proof. It is easy to verify that the function next is σ-continuous on $\{x\} \cup B(x)$ and has no fixed point in $B(x)$. Clearly k_x is σ-continuous if and only if

$$\forall S \in F: S \subseteq C(x) \cup B(x), \text{next}(\text{glb}(\text{lub } S, x)) = \text{lub}(\text{next}(\text{glb}(S, x))).$$

Therefore, as $\text{lub}(\text{next}(\text{glb}(S, x))) = \text{next}(\text{lub}(\text{glb}(S, x)))$ the condition becomes

$$\text{next}(\text{glb}(\text{lub } S, x)) = \text{next}(\text{lub}(\text{glb}(S, x))).$$

In general $\text{lub}(\text{glb}(S, x)) \leq \text{glb}(\text{lub } S, x)$.

If $\text{glb}(\text{lub}(S, x)) = \text{lub}(\text{glb}(S, x))$, the condition is clearly satisfied. If $\text{lub}(\text{glb}(S, x)) < \text{glb}(\text{lub } S, x)$, then $\forall y, z \in B(x)$ $y < z$ implies $\text{next}(y) < \text{next}(z)$ so the condition is satisfied if and only if

$$\text{glb}(\text{lub}(S, x)) = x \quad \text{and} \quad \text{next}(\text{lub}(\text{glb}(S, x)) = x.$$

But $\text{next}(\text{lub}(\text{glb}(S, x))) = x$ if and only if $B(x)$ is finite and $\text{lub}(\text{glb}(S, x))$ is the greatest element of $B(x)$.

From the previous lemmas and propositions we can derive the general lemma

Lemma 3.7. *Let D be a complete lattice F a σ-family, $x \in D$, Y_σ the least fixed point operator, $Y_\sigma^{-1}(x)$ the set of σ-continuous functions in $D \to D$ which have x as a least fixed point, then $Y_\sigma^{-1}(x)$ has a least element if and only if*

(i) *$B(x)$ is an ascending sequence and*

(ii) *$\forall S \in F$ $S \subseteq C(x) \cup B(x)$, either* $\text{lub}(\text{glb}(S, x)) = \text{glb}(\text{lub } S, x)$ *or $B(x)$ is finite,* $x = \text{glb}(\text{lub } S, x)$ *and* $\text{lub}(\text{glb}(S, x))$ *is equal to the greatest element of $B(x)$.*

If we assume that D is σ-normal the second condition is trivially verified and we may state

Theorem 3.8. *If D is a complete σ-normal lattice, then for any $x \in D$ $Y_\sigma^{-1}(x)$ has a least element if and only if $B(x)$ is an ascending sequence.*

A similar theorem can be proved for monotonic functions. The conditions related to σ-continuity are no more necessary but $B(x)$ has to be a well-ordered chain (every element of $B(x)$ has an immediate successor in $B(x)$).

Theorem 3.9. *Let D be a complete lattice and x an element of D. The set of monotonic functions which have x as a least fixed point has a least element if and only if $B(x)$ is a well-ordered chain.*

4. Characterization of the minimal elements of $Y_\sigma^{-1}(x)$

The preceding relation between functions and ascending sequences can be extended.

Theorem 4.1. *Let D be a σ-normal complete lattice and $x \in D$. There is a one-to-one correspondence between the maximal ascending sequences converging to x and the minimal elements of $Y_\sigma^{-1}(x)$.*

This theorem is a consequence of the following propositions that make this correspondence explicit. We assume that D is a σ-normal complete lattice, x an element of D and $Y_\sigma^{-1}(x)$ the set of σ-continuous functions that have x as least fixed point. F is the σ-family.

Definition. A non-empty subset S_1 of a subset S of a lattice D is an *upper part* of S if and only if

$$\forall t_1 \in S_1, \forall t \in S: t \geqslant t_1 \Rightarrow t \in S_1.$$

Lemma 4.2. *Let f be a monotonic function and S a directed set, T an upper part of S, then*

$$f(\text{lub } T) = \text{lub } f(T) \textit{ if and only if } f(\text{lub } S) = \text{lub } f(S).$$

Proposition 4.3. *If $g \in Y_\sigma^{-1}(s)$ is minimal, then the sequence $\{g^i(\perp)\}_{i \in N}$ is a maximal ascending sequence converging to x.*

Proof. If $\{g^i(\perp)\}_{i \in N}$ is not maximal, then there exist $y \in D$, $i \in N: g^i(\perp) < y < g^{i+1}(\perp)$.

Define the function h:

$$\forall d \in D, h(d) = \begin{cases} \text{glb}(y, g(d)) & \text{if } d \leqslant g^i(\perp), \\ \text{glb}(g(d), g^{i+1}(\perp)) & \text{if } d \leqslant y, d \nleqslant g^i(\perp), \\ g(d) & \text{if } d \nleqslant y. \end{cases}$$

Clearly, h is monotonic and $h < g$ We first show that h is σ-continuous, then that it admits x as a least fixed point.

Let $S \in F$. Consider the three cases:

(1) there exist d in S such that $d \nleqslant y$,

(2) $\forall d \in S$, $d \leqslant y$ but there exists $d \in S$: $d \nleqslant g^i(\perp)$,

(3) $\forall d \in S$, $d \leqslant g^i(\perp)$.

Case 1: Let $T = \{d \in S \mid d \nleqslant y\}$. T is an upper part of S.

$$\forall t \in T: h(t) = g(t),$$

$$h(\text{lub } T) = g(\text{lub } T) = \text{lub } g(T) = \text{lub } h(T)$$

and so

$$\text{lub } h(S) = h(\text{lub } S).$$

Case 2: Define $T=\{d\in S\,|\,d\not\leqslant g^i(\perp)\}$, T is an upper part of S. We have

$$\operatorname{lub} h(T)=\operatorname{lub}(\operatorname{glb}(g(T), g^{i+1}(\perp)))=\operatorname{glb}(\operatorname{lub}(g(T)), g^{i+1}(\perp))$$
$$=\operatorname{glb}(g(\operatorname{lub} T), g^{i+1}(\perp))=h(\operatorname{lub} T)$$

since clearly lub $T\leqslant y$ and lub $T\not\leqslant g^i(\perp)$. Therefore lub $h(S)=h(\operatorname{lub} S)$.

Case 3: We have similarly

$$\operatorname{lub} h(S)=\operatorname{lub}(\operatorname{glb}(y, g(S)))=\operatorname{glb}(y, \operatorname{lub} g(S))$$
$$=\operatorname{glb}(y, g(\operatorname{lub} S))=h(\operatorname{lub} S)$$

since lub $S\leqslant g^i(\perp)$.

Now we have to show that h has x as the least fixed point. On $A(x)$ and $C(x)$, h is identical to g, so x is a fixed point of h and h has no fixed point in $C(x)$. So we have to show that h has no fixed point in $B(x)$, which amounts to proving

$$d\neq \operatorname{glb}(y, g(d)) \quad \text{if } d\leqslant g^i(\perp)$$

and

$$d\neq \operatorname{glb}(g(d), g^{i+1}(\perp)) \quad \text{if } d\leqslant y \text{ and } d\not\leqslant g^i(\perp).$$

In the first case there exists a largest j such that: $g^j(\perp)\leqslant d$, $g^{j+1}(\perp)\leqslant g(d)$ and $g^{j+1}(\perp)\leqslant y$. So $g^{j+1}(\perp)\leqslant \operatorname{glb}(y, g(d))=h(d)$, as $g^{j+1}(\perp)\not\leqslant d$, d cannot be equal to $h(d)$. Similarly for the second case. Therefore g is not minimal and the proof is concluded.

Proposition 4.4. *Let $\{y_i\}_{i\in N}$ be a maximal ascending sequence converging to x. The function g defined below is a minimal element of $Y_\sigma^{-1}(x)$. Furthermore g is the least function in the set of functions*:

$$\{f: D\to D\,|\,f \text{ monotonic and } f^i(\perp)=y_i\ \forall i\in N\},$$

$$\forall d\in D,\ g(d)=\begin{cases} x & \text{if } d\in A(x),\\ y_{i+1} & \text{otherwise, where } i \text{ is the largest number such that } y_i\leqslant d.\end{cases}$$

Proof. Clearly g is monotonic and x is its least fixed point. One also verifies easily that, by construction, g is the least monotonic function f such that $f^i(\perp)=y_i\ \forall i\in N$. Now if $h\in Y_\sigma^{-1}(x)$ and $h\leqslant g$, a simple induction shows that $h^i(\perp)=g^i(\perp)=y_i\ \forall i\in N$, otherwise $\{y_i\}_{i\in N}$ would not be maximal. As the σ-continuous functions are also monotonic, we find $h=g$. It remains to show that g is σ-continuous.

Let i be the largest integer such that $y_i\leqslant d$ for some given $d\in B(x)\cup C(x)$. Clearly $y_i=\operatorname{glb}(y_{i+1}, d)$, otherwise $\{y_i\}_{i\in N}$ would not be maximal.

Assume now that $g(\operatorname{lub} S)=y_k$ for some $S\in F$, $S\subseteq B(x)\cup C(x)$. We have then

$$y_{k-1}=\operatorname{glb}(g(\operatorname{lub} S), \operatorname{lub} S)=\operatorname{lub}(\operatorname{glb}(y_k, S)).$$

Consider the sequence of sets of

$$S_1 = \text{glb}(y_k, S),$$
$$S_2 = \text{glb}(y_{k-2}, S_1),$$
$$S_3 = \text{glb}(y_{k-3}, S_2),$$
$$\vdots$$
$$S_{k-1} = \text{glb}(y_1, S_{k-2}).$$

We have

$$\text{lub}\, S_1 = y_{k-1},$$
$$\text{lub}\, S_2 = \text{lub}(\text{glb}(y_{k-2}, S_1)) = \text{glb}(y_{k-2}, \text{lub}\, S_1) = y_{k-2}.$$

Similarly

$$\text{lub}\, S_p = y_{k-p}, \qquad \text{lub}\, S_{k-1} = y_1.$$

As $\{y_i\}_{i\in N}$ is maximal, S_{k-1} is reduced to $\{y_1\}$ or $\{\perp, y_1\}$ and so $y_1 \in S_{k-1}$, which implies that there exists $d \in S_{k-2}$: $y_1 = \text{glb}(y_1, d)$. d can only be equal to y_2, otherwise $\{y_i\}_{i\in N}$ would not be maximal. Therefore $y_2 \in S_{k-2}$, and a straightforward induction leads to $y_{k-1} \in S_1$. That is, there exists d in S such that $y_{k-1} = \text{glb}(y_k, d)$ therefore

$$y_k = g(d) \leqslant \text{lub}\, g(S) \leqslant g(\text{lub}\, S) = y_k$$

and we have the desired equality

$$g(\text{lub}\, S) = \text{lub}\, g(S).$$

Assume now that $g(\text{lub}\, S) = x$. This implies $\text{lub}\, S \geqslant y_i\ \forall i \in N$ by definition of g. Therefore

$$\forall i \in N: y_i = \text{glb}(\text{lub}\, S, y_i) = \text{lub}(\text{glb}(S, y_i)).$$

A construction similar to the preceding one leads us to conclude that there exists $d \in S$: $y_i = \text{glb}(d, y_i)$, that is, $\forall i \in N$ there exists $d \in S$: $y_i \leqslant d$. Clearly then $\text{lub}\, g(S) = x = g(\text{lub}\, S)$.

Here again we can generalize to monotonic functions. The proof is similar to the preceding one and uses Hitchcock and Park's generalized fixed point theorem [3].

Theorem 4.5. *Let D be a complete lattice and x an element of D. There exists a one to one correspondence between the set of minimal monotonic functions having x as a least fixed point and the set of maximal well-ordered chains converging to x.*

Now the following theorem shows that we can restrict ourselves to increasing ($f(t) \geqslant t\ \forall t \in D$) σ-continuous functions to obtain the maximal ascending sequences in a σ-normal complete lattice D. It may also be interesting to note that the set of increasing σ-continuous functions which have x as a least fixed point forms a monoid.

Theorem 4.6. *Let D be a σ-normal complete lattice. There is a one to one correspondence between the set of maximal ascending sequences converging to $x \in D$ and the set of minimal increasing σ-continuous functions having x as a least fixed point.*

Proof. The proof follows the one given for Theorem 4.1 to show that if g is a minimal increasing σ-continuous function, then $\{g^i(\perp)\}_{i \in N}$ is a maximal ascending sequence. Conversely if $\{y_i\}_{i \in N}$ is a maximal ascending sequence converging to x, define f as follows, which has x as a least fixed point

$$f(d) = \begin{cases} d & \text{if } d \in A(x), \\ \text{lub}(d, y_{i+1}) & \text{otherwise, where } i \text{ is the largest number such that } y_i \leqslant d. \end{cases}$$

f is clearly minimal, monotonic and increasing. We have to show that f is σ-continuous. Let S be an element of the underlying σ-family.

Suppose $f(\text{lub}\, S) = \text{lub}(\text{lub}\, S, y_{i+1})$, then as in the proof of Theorem 4.1, there exists an element d in S such that $y_i = \text{glb}(y_{i+1}, d)$, that is, $y_i \leqslant d$. Let $T = \{t \in S \mid t \geqslant d\}$, then clearly $\text{lub}\, T = \text{lub}\, S$. Now

$$\text{lub}\, f(T) = \text{lub}\{\text{lub}(T, y_{i+1})\} = \text{lub}\{\text{lub}\, T, y_{i+1}\} = f(\text{lub}\, T)$$

and therefore $\text{lub}\, f(S) = f(\text{lub}\, S)$.

Suppose $f(\text{lub}\, S) = \text{lub}\, S$, then $\text{lub}\, S \in A(x)$. If there exists $d \in S$ such that $d \in A(x)$, it is trivial. Else $\text{lub}\, f(S) = \text{lub}\{\text{lub}(d, y(d)), d \in S\}$ where $y(d)$ represents the appropriate y_{i+1} corresponding to the element d. One verifies easily: $\text{lub}\, f(S) = \text{lub}\{\text{lub}\, S, \text{lub}(y(d), d \in S)\}$.

As $\text{lub}\, S \in A(x)$ one finds $\text{lub}(y(d), d \in S) = x$, since, as in the proof of Theorem 4.1, for every y_i there exists $d \in S: d \geqslant y_i$, and $\text{lub}\, f(S) = \text{lub}\{\text{lub}\, S, x\} = \text{lub}\, S = f(\text{lub}\, S)$.

Corollary 4.7. *For a σ-normal complete lattice D and for every $x \in D$ there is a one to one correspondence between minimal σ-continuous functions having x as a least fixed point and minimal increasing σ-continuous functions having x as a least fixed point.*

More generally using the construction described below one can show easily that if x is the least fixed point of a σ-continuous function f, then it is the least fixed point of an increasing σ-continuous function g having the same ascending sequence:

$$g(d) = \begin{cases} f^{i+1}(\perp) & \text{when } d \leqslant f^i(\perp) \text{ and } d \not\leqslant f^j(\perp) \text{ for any } j < i, \\ \text{lub}(x, d) & \text{otherwise.} \end{cases}$$

This construction is different from the one given in the Theorem 4.6 which gives a σ-continuous function only when f is minimal, but has the advantage of leading to a one to one correspondence.

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